

Close Tue: 14.3(2), 14.4

Close Thur: 14.7

Reminder of Extra Help Options

Office Hours - 12:30-2:00pm (PDL C-339)

Math Tutors - 9:30am-9:30pm (Com B-014)

CLUE Tutors - 7:00pm - midnight (Mary Gates)

## 14.4/14.7 Tangent Planes & Max/Min

Remember from Math 124:

The equation of the tangent line to the curve  $y = f(x)$  at  $x_0$  is given by

$$y - f(x_0) = f'(x_0)(x - x_0)$$

Similarly, the equation of the tangent plane to the surface  $z = f(x, y)$  at  $(x_0, y_0)$  is given by

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

## Entry Tasks:

1. Find the equation for the tangent

line to  $f(x) = x^3$  at  $x = 2$ .

$$f(2) = (2)^3 = 8 = \text{height}$$

$$f'(x) = 3x^2$$

$$f'(2) = 3(2)^2 = 12 = \text{slope of tangent}$$

$$y - 8 = 12(x - 2)$$

TANGENT  
LINE

2. Find the equation for the tangent

plane to  $f(x, y) = x^2 + 3y^2x - y^3$  at  $(x, y) = (2, 1)$ .

$$f(2, 1) = (2)^2 + 3(1)^2(2) - (1)^3 = 4 + 6 - 1 = 9 = \text{HEIGHT}$$

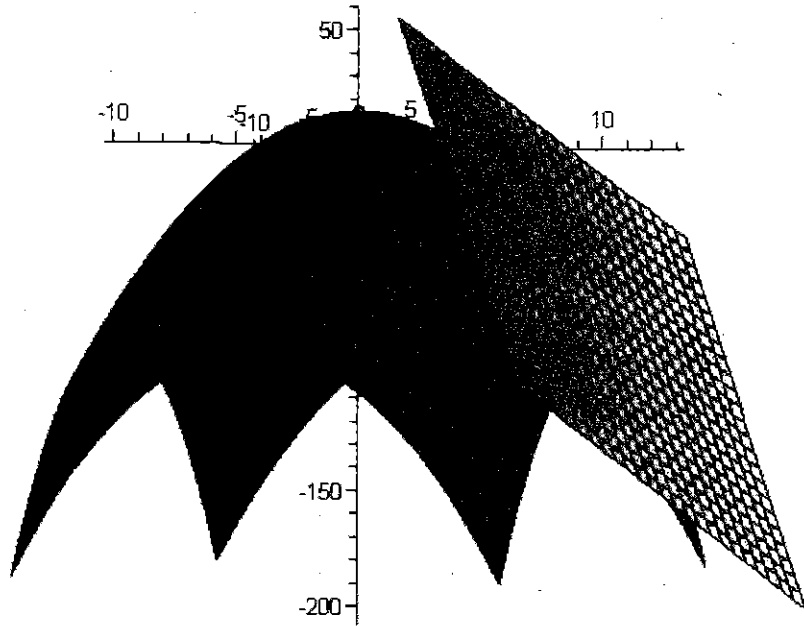
$$f_x = 2x + 3y^2 \Rightarrow f_x(2, 1) = 4 + 3 = 7 = \text{slope of tangent in } x\text{-direction}$$

$$f_y = 6yx - 3y^2 \Rightarrow f_y(2, 1) = 12 - 3 = 9 = \text{slope of tangent in } y\text{-direction}$$

$$z - 9 = 7(x - 2) + 9(y - 1)$$

TANGENT PLANE

## Visuals and Derivation of Tangent Plane



TANGENT PLANE

$$z - (-50) = -14(x - 7) - 8(y - 4)$$

$$z = f(x, y) = 15 - x^2 - y^2 \text{ at } (7, 4)$$

Recall:

$$f(7, 4) = -50$$

$$f_x(7, 4) = -14$$

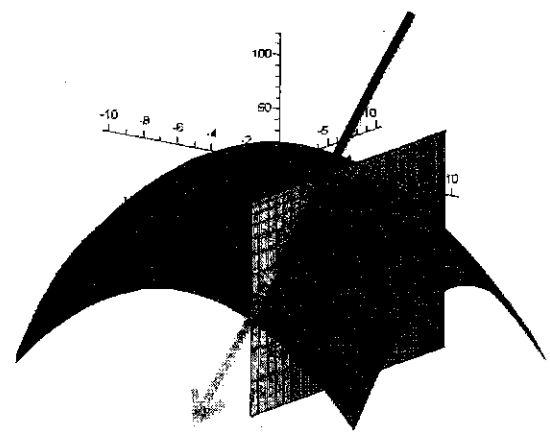
$$f_y(7, 4) = -8$$

## Derivation of Tangent Plane

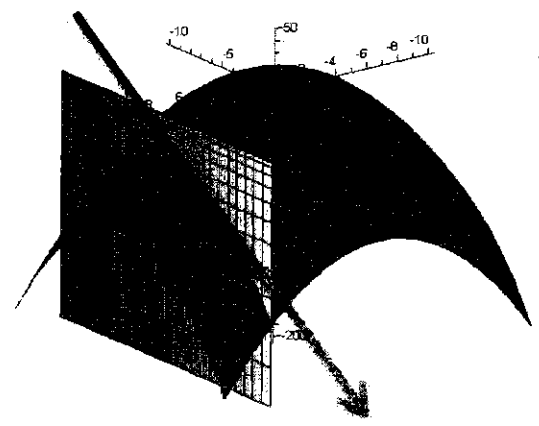
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The plane goes thru (7, 4, -50).  
Now we need a normal vector.

Note:  
 $f_x(x,y) = -2x$   
 $f_x(7,4) = -14$



$f_y(x,y) = -2y$   
 $f_y(7,4) = -8$



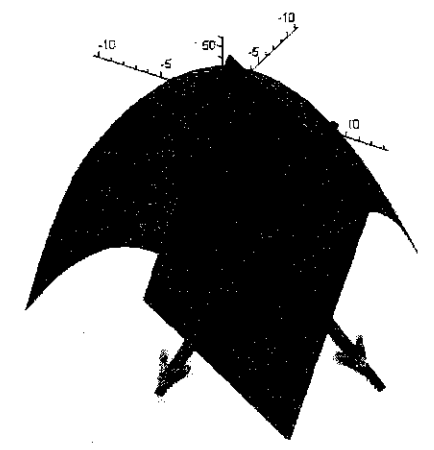
$$\square(x - \square) + \square(y - \square) + \square(z - \square) = 0$$

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Thus, we can get two vectors that are parallel to the plane:

$$\langle 1, 0, f_x(x_0, y_0) \rangle = \langle 1, 0, -14 \rangle$$
$$\langle 0, 1, f_y(x_0, y_0) \rangle = \langle 0, 1, -8 \rangle$$

So a normal vector is given by  
 $\langle 1, 0, -14 \rangle \times \langle 0, 1, -8 \rangle = \langle 14, 8, 1 \rangle$



**Tangent Plane:**

$$14(x-7) + 8(y-4) + (z+50) = 0$$

Which we rewrite as:

$$z + 50 = -14(x-7) - 8(y-4)$$

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**Linear Approximation:** "Near" the point  $(x_0, y_0)$  the tangent plane and surface have z-values that are close together.

Take  $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ ,  
add  $z_0$  to both sides to get

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

We call the above function,  $L(x, y)$ , the linear approximation to  $f(x, y)$  at  $(x_0, y_0)$ .

**Idea:**  $f(x, y) \approx L(x, y)$  for  $(x, y) \approx (x_0, y_0)$

**Example:** Use the linear approximation to  $f(x, y) = x^2 + 3y^2x - y^3$  at  $(x, y) = (2, 1)$  to estimate the value of  $f(1.9, 1.05)$ .

$$\underbrace{x^2 + 3y^2x - y^3}_{f(x,y)} \approx \underbrace{9 + 7(x-2) + 9(y-1)}_{L(x,y)}$$

$$L(1.9, 1.05) = 9 + 7(1.9 - 2) + 9(1.05 - 1) = 9 - 0.7 + 0.45 = 8.75 \leftarrow \text{linear approximation}$$

$$f(1.9, 1.05) = (1.9)^2 + 3(1.05)^2(1.9) - (1.05)^3 \approx 8.736625 \leftarrow \text{ACTUAL VALUE.}$$

## 14.7 Local Max/Min

Consider the surface  $z = f(x,y)$ .

*Some Terminology:*

A **local maximum** occurs at  $(a,b)$  if  $f(a,b)$  is larger than *all* values "near" it (top of a hill).

A **local minimum** occurs at  $(a,b)$  if  $f(a,b)$  is smaller than *all* values "near" it (bottom of a valley).

A **critical point** is a point  $(a,b)$  where **BOTH**

$f_x(a,b) = 0$  AND  $f_y(a,b) = 0$  or where either partial doesn't exist.

If  $f_x(a,b) = 0$  and  $f_y(a,b) = 0$ , and  $(a,b)$  is not a local max or min, then we call it a **saddle point**.

*Example:* Find the critical points of

$$f(x,y) = 3xy - \frac{1}{2}y^2 + 2x^3 + \frac{9}{2}x^2$$

$$\boxed{1} \quad f_x = 3y + 6x^2 + 9x \stackrel{?}{=} 0$$

$$\boxed{2} \quad f_y = 3x - y \stackrel{?}{=} 0$$

REMINDER OF HOW TO SOLVE A SYSTEM OF EQUATIONS.

**STEP 1** Solve for one variable in one equation (whichever is easier)

$$\boxed{2} \Rightarrow 3x = y$$

**STEP 2** Substitute into other equation

$$\boxed{2} \text{ \& \ } \boxed{1} \Rightarrow 3(3x) + 6x^2 + 9x \stackrel{?}{=} 0$$
$$6x^2 + 18x = 0$$

**STEP 3** Solve!

$$6x(x+3) = 0$$

$$x=0 \quad \text{or} \quad x+3=0$$
$$x=-3$$

Two Answers

$$x=0 \Rightarrow y=3(0)=0 \Rightarrow (x,y) = (0,0)$$

$$x=-3 \Rightarrow y=3(-3)=-9 \Rightarrow (x,y) = (-3,-9)$$

CHECK!

Example: Find and classify all critical points for

$$f(x, y) = 3xy - \frac{1}{2}y^2 + 2x^3 + \frac{9}{2}x^2$$

$$f_{xx} = 12x + 9$$

$$f_{yy} = -1$$

$$f_{xy} = 3 = f_{yx}$$

$$D = f_{xx} f_{yy} - \underbrace{f_{xy} f_{yx}}_{(f_{xy})^2} = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

$$D = (12x + 9)(-1) - (3)^2$$

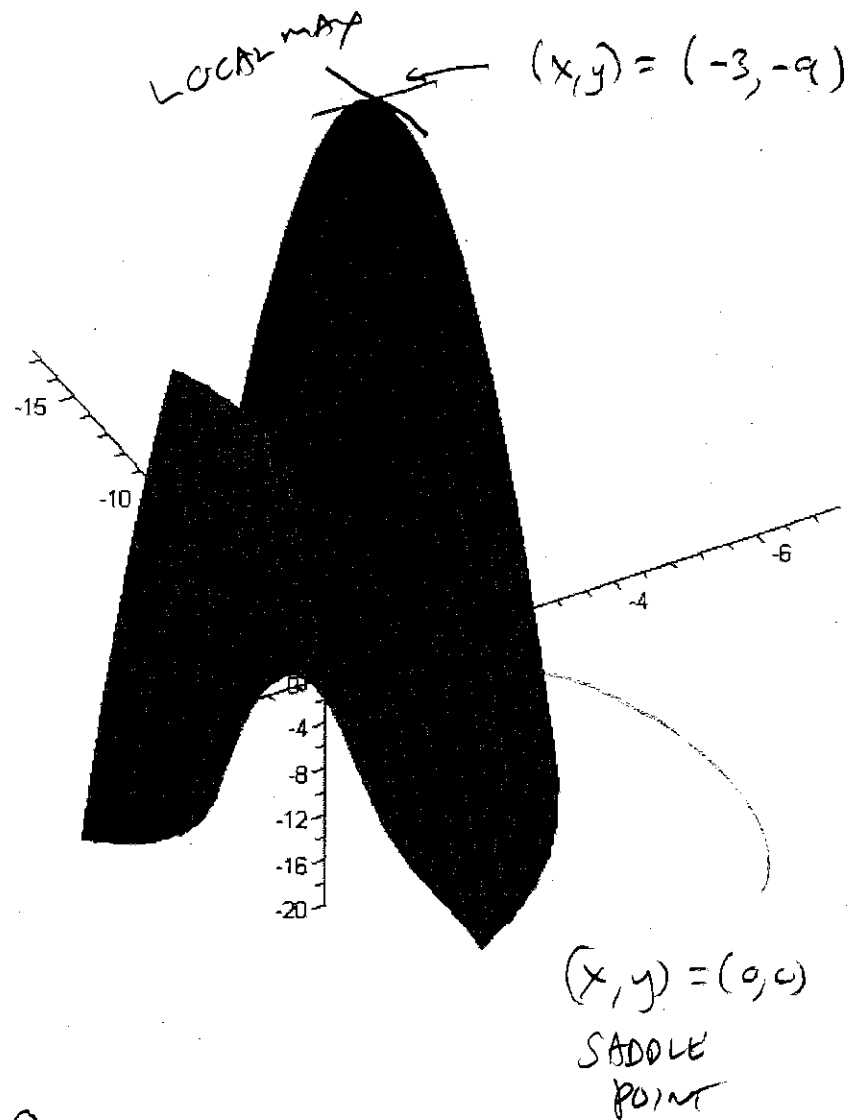
$$(0, 0) \Rightarrow D = (9)(-1) - 9 = -18 < 0$$

$$(-3, -9) \Rightarrow D = \underbrace{(-36 + 9)}_{-27} \underbrace{(-1)}_{-1} - 9 = 27 - 9 = 18 > 0$$

ASIDE: LOCAL MAX VALUE = OUTPUT =  $f(-3, -9) =$

$$3(-3)(-9) - \frac{1}{2}(-9)^2 + 2(-3)^3 + \frac{9}{2}(-3)^2$$

$$= \boxed{27} = \text{LOCAL MAX}$$



## Second Derivative Test

Let  $(a,b)$  be a critical point.

Find all **second** partials at  $(a,b)$

and compute

$$D = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

1. If  $D > 0$ , then the concavity is the same in all directions. So

(a) If  $f_{xx} > 0$ , then it is concave up in all directions. **Local Minimum.**  $\Rightarrow f_{yy} > 0$  As well

(b) If  $f_{xx} < 0$ , then it is concave down in all directions. **Local Maximum.**  $\Rightarrow f_{xx} < 0$  As well

2. If  $D < 0$ , then the concavity changes in some direction. **Saddle Point.**

3. If  $D = 0$ , the test is **inconclusive**.  
(need a contour map)

*Quick Examples:*

**1.  $f(x,y) = 15 - x^2 - y^2$ ,**

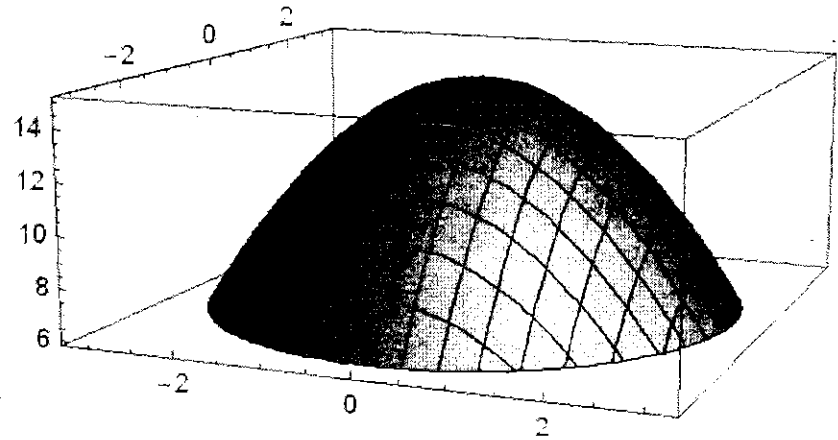
Critical pt: (0,0).

$$f_{xx} = -2, f_{yy} = -2, f_{xy} = 0$$

$$D = (-2)(-2) - (0)^2 = 4$$

$$D > 0, f_{xx} < 0, f_{yy} < 0$$

Local max!



**2.  $f(x,y) = x^2 + y^2$ ,**

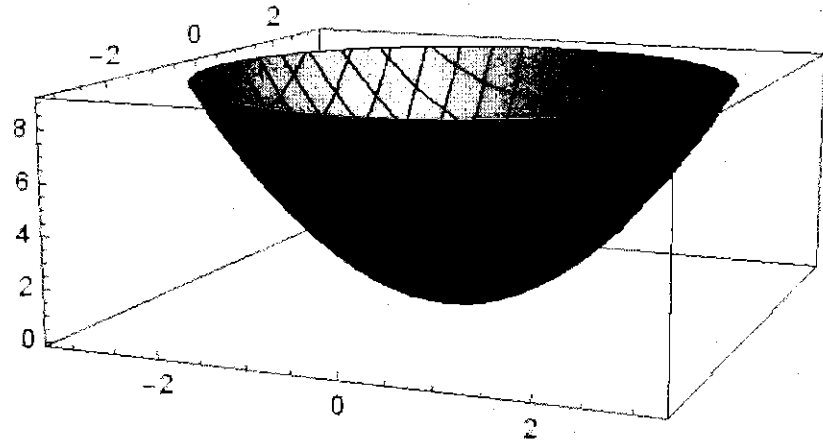
Critical pt: (0,0).

$$f_{xx} = 2, f_{yy} = 2, f_{xy} = 0,$$

$$D = (2)(2) - (0)^2 = 4$$

$$D > 0, f_{xx} > 0, f_{yy} > 0$$

Local min!



**3.  $f(x,y) = x^2 - y^2$**

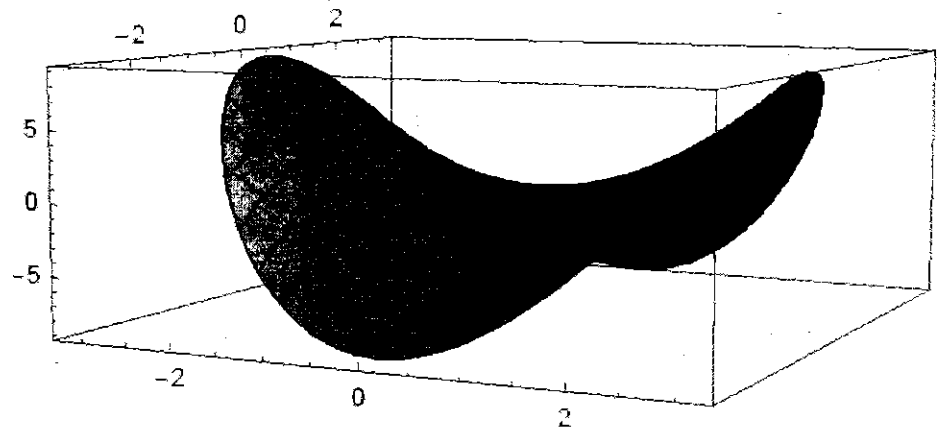
Critical pt: (0,0).

$$f_{xx} = 2, f_{yy} = -2, f_{xy} = 0,$$

$$D = (2)(-2) - (0)^2 = -4$$

$$D < 0 \text{ (note also, } f_{xx} < 0, f_{yy} > 0)$$

Saddle point!





Examples from old exams:

1. Find and classify all critical points for

$$f(x, y) = x^2 + 4y - x^2y + 1$$

$$\textcircled{1} \quad 2x - 2xy \stackrel{?}{=} 0$$

$$\textcircled{2} \quad 4 - x^2 \stackrel{?}{=} 0$$

$$\textcircled{1} \Rightarrow 2x(1-y) = 0 \Rightarrow x=0 \text{ or } y=1$$

$$\textcircled{2} \Rightarrow x^2 = 4 \Rightarrow x = -2 \text{ or } x = 2$$

WHAT WORKS IN BOTH!!!

$x=0$  DOES NOT WORK IN BOTH.

$x=-2, y=1$  DOES WORK IN BOTH

$x=2, y=1$  DOES WORK IN BOTH

|           |          |
|-----------|----------|
| $(-2, 1)$ | $(2, 1)$ |
|-----------|----------|

$$f_{xx} = 2 - 2y, \quad f_{yy} = 0, \quad f_{xy} = -2x$$

$$D = f_{xx}f_{yy} - (f_{xy})^2 = 0 - 4x^2$$

AT  $(-2, 1)$   $D < 0$  SADDLE POINT

AT  $(2, 1)$   $D < 0$  SADDLE POINT

2. Find and classify all critical points for

$$f(x, y) = \frac{9}{x} + 3xy - y^2$$

$$\textcircled{1} \quad f_x = -\frac{9}{x^2} + 3y \stackrel{?}{=} 0$$

$$\textcircled{2} \quad f_y = 3x - 2y \stackrel{?}{=} 0$$

$$\textcircled{2} \Rightarrow 3x = 2y \Rightarrow y = \frac{3}{2}x$$

$$\textcircled{1} + \textcircled{2} \Rightarrow -\frac{9}{x^2} + 3\left(\frac{3}{2}x\right) \stackrel{?}{=} 0$$

$$-\frac{9}{x^2} + \frac{9}{2}x = 0 \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{MULT. BY } x^2$$

$$-9 + \frac{9}{2}x^3 = 0$$

$$\frac{9}{2}x^3 = 9$$

$$x^3 = 2$$

$$x = (2)^{1/3}$$

$$\Rightarrow y = \frac{3}{2}(2)^{1/3}$$

|                                   |
|-----------------------------------|
| $(2^{1/3}, \frac{3}{2}(2)^{1/3})$ |
|-----------------------------------|

$$f_{xx} = \frac{18}{x^3}, \quad f_{yy} = -2, \quad f_{xy} = 3$$

$$D = \frac{18}{x^3}(-2) - (3)^2$$

For  $x = 2^{1/3}$   $D < 0$

|              |
|--------------|
| SADDLE POINT |
|--------------|

3. Find and classify all critical points for

$$f(x, y) = x^2y - 9y - xy^2 + y^3$$

$$\text{① } f_x = 2xy - y^2 \stackrel{?}{=} 0$$

$$\text{② } f_y = x^2 - 9 - 2xy + 3y^2 \stackrel{?}{=} 0$$

$$\text{①} \Rightarrow y(2x - y) = 0$$

$$\Rightarrow y = 0 \quad \text{OR} \quad \underbrace{2x - y = 0}_{2x = y}$$

TWO CASES

$$\text{i) } y = 0 \Rightarrow \text{② } \left. \begin{array}{l} x^2 - 9 - 0 + 0 = 0 \\ \Rightarrow x = \pm 3 \end{array} \right\} \begin{array}{l} (-3, 0) \\ (3, 0) \end{array}$$

$$\text{ii) } y = 2x \Rightarrow \text{② } \left. \begin{array}{l} x^2 - 9 - 2x(2x) + 3(2x)^2 \stackrel{?}{=} 0 \\ x^2 - 9 - 4x^2 + 3 \cdot 4x^2 \stackrel{?}{=} 0 \end{array} \right\}$$

$$-9 + 9x^2 = 0$$

$$9x^2 = 9$$

$$x^2 = 1$$

$$x = \pm 1$$

$$\Rightarrow y = 2x = \pm 2$$

$$f_{xx} = 2y, \quad f_{yy} = -2x + 6y$$

$$f_{xy} = 2x - 2y$$

$$(-3, 0) \Rightarrow f_{xx} = 0, f_{yy} = 6, f_{xy} = -6,$$

$$D = (0)(6) - (-6)^2 < 0$$

SADDLE POINT

$$(3, 0) \Rightarrow f_{xx} = 0, f_{yy} = -4, f_{xy} = 6$$

$$D = (0)(-4) - (6)^2 < 0$$

SADDLE POINT

$$(-1, -2) \Rightarrow f_{xx} = -4, f_{yy} = -10$$

$$f_{xy} = 2$$

$$D = 40 - (2)^2 = 36 > 0$$

LOCAL MAX

$$(1, 2) \Rightarrow f_{xx} = 4, f_{yy} = 10$$

$$f_{xy} = -2$$

$$D = 40 - (-2)^2 = 36 > 0$$

$$(-1, -2)$$

$$(1, 2)$$

LOCAL MIN